ASYMPTOTIC SOLUTIONS FOR THE FLOW OF A NON-NEWTONIAN THIRD GRADE FLUID IN AN ORTHOGONAL RHEOMETER

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ABSTRACT

The flow of a third grade fluid in an orthogonal rheometer is studied. The admissible velocity field used, is proposed by Rajagopal. The flow problem is solved and the velocity field is obtained by considering an asimptotic development of the solutions in respect to a small parameter. The zero and one order approximations of the unknown functions that define the velocity field are determinated. Some diagrams concerning the velocityprofiles are presented.

KEYWORDS: *orthogonal rheometer, asymptotic solutions, third grade fluid.*

1. INTRODUCTION

The apparatus called "orthogonal rheometer" has two parallel plates rotating with the same constant angular velocity Ω, around two parallel and distinct axes and the fluid to be tested fills the space between them.

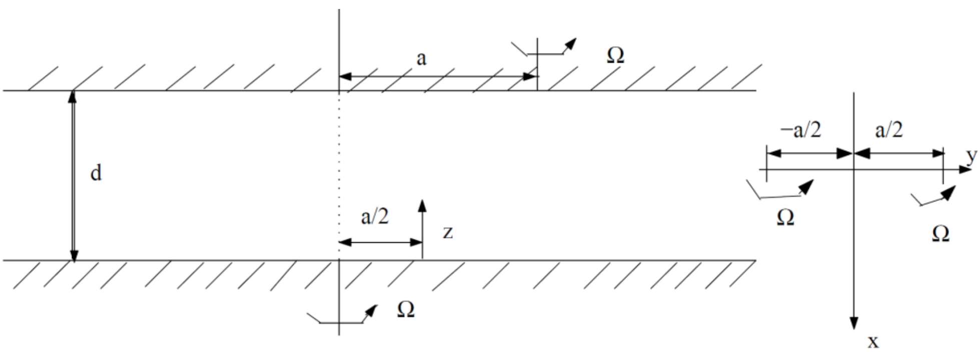

Figure 1. Orthogonal Rheometer.

Elements:

· *a* = the distance between the two rotational axes
· *d* = the distance between the plates
· Ω = the angular velocity

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The motion occurring in the orthogonal rheometer has been studied by several authors. Abbot and Walters in [1], obtained an exact solution for the linear viscous incompressible fluid. They assume that the end effects can be neglected and that the flow can be represented as if the boundaries are two infinite parallel plates. They did not assume the distance between the axes small or inertial effects negligible. Then assuming the distance between the axes small, they studied the flow of a viscoelastic fluid in the same domain by means of an expansions of a series of power in respect to a small parameter.

Huilgol in [2], introduced for the first time the following velocity field:

$$u = -\Omega(y - \psi(z)),\ v = \Omega x,\ w = 0,$$

In this formula *u*, *v* and *w*, are the components of velocity in the direction of *x*, *y* and z. This field satisfies the equation of motion of a simple fluid, provided inertial effects are negligible. If the inertial terms are not ignored, the balance principles are not satisfied.

Rajagopal and Gupta in [3] and Rajagopal in [4] obtained an exact solution of the flow of an homogeneous incompressible fluid of seconds grade in the same domain, without presuming that the distance between axes is small. However, the specific form introduced by Rajagopal and Gupta in [3], reduces the equations of motion to the same degree as the ones for the linear viscous fluid, therefore, the adherence conditions to the boundary became sufficient to determine a unique solution.

In [5], Rajagopal studied the flow of a simple fluid in the same domain, without neglecting the inertial effects and the adherence conditions to the boundary were satisfactory for determine the solution. Considering the flow an incompressible fluid of second grade, the boundary value problem can be solved even when the inertial effects are included.

Rajagopal and Wineman in [6], studied the flow of a BKZ fluid in the case of the linear vascoelasticity. They proved in this paper that for reasonable operating conditions of the reomether, the effects of inertia are very small

In [7], Tigoiu and Niculescu studied the flow of a second order fluid in an orthogonal rheometer. With the help of an asymptotic development in respect to the Weissemberg number, they proved the existence and uniqueness of the solutions for approximate solutions of the flow problem with boundary conditions.

Pricina, Tigoiu, and Cipu in [8] studied the flow in the orthogonal rheometer of the BKZ and Wagner fluids. Regarding the BKZ fluid, they used Currie's potential (see [9]). The solution of the problem was obtained by the help of the asymptotic development in respect to a small parameter. For the Wagner fluid, they solve the problem in the case of slow motions.

Rajagopal in [5], using the velocity field proposed by Rajagopal and Gupa in [3]:

$$\begin{gathered} u = -\Omega(y - g(z)), \\ v = \Omega(x - f(z)), \\ w = 0, \end{gathered} \tag{1}$$

showed that the motion is one of constant stretch history. This type of motions was thoroughly analyzed by Coleman, Markovitz and Noll in [10].

2. EQUATIONS OF THE MOTION

In this part are presented the results obtained by Rajagopal in [5]. He studies the flow of a simple fluid for which the stress tensor T can be express as a function of the first two Rivlin-Eriksen tenors.

The equation of motions is:

$$\rho\vec{a} = \rho\vec{b} + \mathrm{div}\mathbf{T} \tag{2}$$

where *b* is body force and *div* is the divergence operator.

The acceleration for the velocity give by (1) is:

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = -\Omega^2(x - f(z))\vec{i} - \Omega^2(y - g(z))\vec{j} \tag{3}$$

Next, the classical way described by Rajagopal is the one that should be followed.

The steps that should be followed are the following:

We assume that body forces are conservative, which means that they derive from a

potential: $\vec{b} = -\mathrm{grad}\Phi$.

The equation of motion (2) is reduced to:

$$\frac{d\hat{T}_{13}}{dz}\vec{i} + \frac{d\hat{T}_{23}}{dz}\vec{j} + \frac{d\hat{T}_{33}}{dz}\vec{k} = \mathrm{grad}(p + \rho\,\Phi) -$$
$$-\rho\Omega^2[x - f(z)]\vec{i} - \rho\Omega^2[y - g(z)]\vec{j}. \tag{4}$$

where $\vec{i}, \vec{j}, \vec{k}$ are versors in the *x*,*y*,*z* directions.

If it is apply the rotor in the (4), is obtained:

$$\frac{d^2\hat{T}_{13}}{dz^2} = \frac{d}{dz}(\rho\Omega^2 f(z)),$$
$$\frac{d^2\hat{T}_{23}}{dz^2} = \frac{d}{dz}(\rho\Omega^2 g(z)). \tag{5}$$

Integrating the system (5), it gets:

$$\frac{d\hat{T}_{13}}{dz} = \rho\Omega^2 f(z) + q,$$
$$\frac{d\hat{T}_{23}}{dz} = \rho\Omega^2 g(z) + s, \tag{6}$$

where *q* and *s* are constants.

In order to guarantee the symmetry of the velocity, they choose:s = q = 0, (according to [5]), we can get unsymmetrical results of the problem, if s and q are not 0)

Therefore, the pressure expression becomes the following:

$$p = \frac{\rho\Omega^2(x^2+y^2)}{2} + \hat{T}_{33} + \rho(C-\Phi) \tag{7}$$

The equations that governing the motion, now become the following:

$$\frac{d\hat{T}_{13}}{dz} = \rho\Omega^2 f(z),$$

$$\frac{d\hat{T}_{23}}{dz} = \rho\Omega^2 g(z). \tag{8}$$

Next, we are going to call the system (8) ,"main system".

The boundary conditions arise from the adherence conditions on the upper and lower plates of the orthogonal rheometer. Since the lower plate is situated at *z=0*, and the upper one at *z=d*, the boundary conditions for velocity field are:

$$u = \frac{\Omega a}{2} - \Omega y,\ v = \Omega x,\ w = 0,\ \text{la } z = d, \tag{9}$$

$$u = -\frac{\Omega a}{2} - \Omega y,\ v = \Omega x,\ w = 0,\ \text{la } z = 0, \tag{10}$$

$$u \to \mp\infty,\ v \to \pm\infty,\ w = 0,\ \text{când } x,\ y \to \pm\infty. \tag{11}$$

From (9) and (10) result the following boundary conditions:

$$f(0) = f(d) = 0,$$

$$g(0) = -a/2,\ g(d) = a/2. \tag{12}$$

Formulating the problem:

Determine the functions *f,g : [0,d] → R* that satisfy equations of motion (8) in (0,d) and the boundary conditions (12).

3. THE FLOW PROBLEM FOR THE THIRD GRADE FLUID

3. 1. Equations **of motion**

The Cauchy's stress tensor **T** for the incompressible fluid of third grade is given by:

$$\mathbf{T} = -p + \hat{\mathbf{T}},\ \hat{\mathbf{T}} = \mu\mathbf{A}_1 + \alpha_1(\mathbf{A}_2 - \mathbf{A}_1^2) + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\text{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{13}$$

where:

· p- hydrostatic pressure

· μ, α_1 $\beta_1, \beta_2, \beta_3$ - constitutive modules (constants) ;

· $A_1, A_2 A_3$ -the first three Rivlin-Ericksen tensors :

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T,$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{L}^T\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L} =$$

$$= \mathbf{L}^T\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L},$$

There are tree constitutive restrictions of the third grade fluid with the constitutive equation (13) (see V. Tigoiu [11]):

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \beta_1 \leq 0,\ \beta_1 + 2(\beta_2 + \beta_3) \geq 0.$$

The velocity field is give by (1) .

The components of the stress tensor **T** are given by:

$$T_{11} \equiv -p + \hat{T}_{11} = -p - \alpha_1\Omega^2 g'^2 - 2\beta_2\Omega^3 f'g',$$

$$T_{12} \equiv \hat{T}_{12} = \alpha_1\Omega^2 f'g' + \beta_2\Omega^3(f'^2 - g'^2),$$

$$T_{13} \equiv \hat{T}_{13} = -\alpha_1\Omega^2 f' + [\mu\Omega - \beta_1\Omega^3 + 2(\beta_2 + \beta_3)\Omega^3(f'^2 + g'^2)]g',$$

$$T_{23} \equiv \hat{T}_{23} = -[\mu\Omega - \beta_1\Omega^3 + 2(\beta_2 + \beta_3)\Omega^3(f'^2 + g'^2)]f' - \alpha_1\Omega^2 g',$$

$$T_{22} \equiv -p + \hat{T}_{22} = -p - \alpha_1\Omega^2 f'^2 + 2\beta_2\Omega^3 f'g',$$

$$T_{33} \equiv -p + \hat{T}_{33} = -p + \alpha_1\Omega^2(f'^2 + g'^2). \tag{14}$$

If we consider the expression of $\hat{T}_{33}$ (see (14)), we obtain from (7) the pressure expression:

$$p = \frac{\rho\Omega^2(x^2 + y^2)}{2} + \alpha_1\Omega^2(f'^2 + g'^2) + \rho(C - \Phi). \tag{15}$$

If we introduce the expressions of the $\hat{T}_{13}$ and $\hat{T}_{23}$ (stress shear components) from (14) in the **“main system”**, we get the following result:

$$\frac{d}{d\,z}\left\{-\alpha_1\Omega^2 f' + [\mu\Omega - \beta_1\Omega^3 + 2(\beta_2+\beta_3)\Omega^3(f'^2+g'^2)]g'\right\} = \rho\Omega^2 f,$$

$$\frac{d}{d\,z}\left\{-[\mu\Omega - \beta_1\Omega^3 + 2(\beta_2+\beta_3)\Omega^3(g'^2+f'^2)]f' - \alpha_1\Omega^2 g'\right\} = \rho\Omega^2 g. \quad (16)$$

We introduce the dimensionless quantities:

$$x = a\overline{x},\quad y = a\overline{y},\quad z = d\overline{z},\quad f = df,\quad g = d\overline{g},\quad p = p_0\bar{p}, \quad (17)$$

where it is clear that the following relations take place:

$$f'(z) = \bar{f}'(\bar{z});\; f''(z) = \frac{1}{d}\bar{f}''(\bar{z}).$$

The dimensionless form of the system (16), is the following:

$$\bar{g}'' - \alpha_m\bar{f}'' + \beta_m[\bar{g}''(\bar{g}'^2+\bar{f}'^2) + 2\bar{g}'(\bar{g}'\bar{g}''+\bar{f}'\bar{f}'')] = Re_m\bar{f},$$

$$-\bar{f}'' - \alpha_m\bar{g}'' - \beta_m[\bar{f}''(\bar{g}'^2+\bar{f}'^2) + 2\bar{f}'(\bar{f}'\bar{f}''+\bar{g}'\bar{g}'')] = Re_m\bar{g}. \quad (18)$$

where the dimensionless parameters are given by the following relations:

$$Re_m = \frac{\rho\Omega d^2}{\mu - \beta_1\Omega^2},$$

$$\alpha_m = \frac{\alpha_1\Omega}{\mu - \beta_1\Omega^2},$$

$$\beta_m = \frac{2\Omega^2(\beta_2+\beta_3)}{\mu - \beta_1\Omega^2}. \quad (19)$$

Formulating the problem:

Determine the functions $\overline{f}, \overline{g} : [0,1] \longrightarrow R$ that satisfy equations of motion

$$\bar{g}'' - \alpha_m\bar{f}'' + \beta_m[\bar{g}''(\bar{g}'^2+\bar{f}'^2) + 2\bar{g}'(\bar{g}'\bar{g}''+\bar{f}'\bar{f}'')] = Re_m\bar{f},$$

$$-\bar{f}'' - \alpha_m\bar{g}'' - \beta_m[\bar{f}''(\bar{g}'^2+\bar{f}'^2) + 2\bar{f}'(\bar{f}'\bar{f}''+\bar{g}'\bar{g}'')] = Re_m\bar{g}, \quad (20)$$

and boundary conditions:

$$\overline{f}(0) = \overline{f}(1) = 0,$$

$$\overline{g}(0) = -a/2d,\quad \overline{g}(1) = a/2d. \quad (21)$$

3.2. The development of the solutions in respect to a small parameter

The bilocal problem we analyze is (20)-(21).

We look to the solution of the above problem in an asymptotic development in respect to a small parameter. We keep in mind the constitutive restrictions obtained by V. Tigoiu in [11].

If in the previous system we used:

$$0 < \beta_m \equiv \frac{2\Omega^2(\beta_2 + \beta_3)}{\mu - \beta_1\Omega^2} << 1,$$

now we can develop the solutions of this system in respect to $\varepsilon \equiv \beta_m << 1$.

We observe that can consider: $0 < 2(\beta_2 + \beta_3)$, because we have the restriction: $\beta_1 + 2(\beta_2 + \beta_3) \geq 0$. Therefore $2(\beta_m + \beta_3) \geq -\beta_1 > 0$ (because $\beta_1 < 0$). From constitutive restrictions we can also see that: $\mu - \beta_1\Omega^2 > 0$.

The development of the $\overline{f}, \overline{g}$ ¯solutions in respect to ε, is the following:

$$\bar{f}(\bar{z}) = \bar{f}_0(\bar{z}) + \varepsilon\bar{f}_1(\bar{z}) + \dots \quad ,$$

$$\bar{g}(\bar{z}) = \bar{g}_0(\bar{z}) + \varepsilon\bar{g}_1(\bar{z}) + \dots \quad . \tag{22}$$

We introduce (22) in the system (20) and identify the coefficients of ε. The system for the zero order approximation (f_0, g_0) is the following:

$$\overline{g}_0'' - \alpha_m \overline{f}_0'' = Re_m \overline{f}_0,$$

$$-\overline{f}_0'' - \alpha_m \overline{g}_0'' = Re_m \overline{g}_0. \tag{23}$$

The boundary conditions for the zero order approximation are:

$$\overline{f}_0(0) = \overline{f}_0(1) = 0,$$

$$\overline{g}_0(0) = -a/2d, \quad \overline{g}_0(1) = a/2d. \tag{24}$$

The above problem (23)-(24), can be solved exactly to yield:

$$\overline{f}_0(\overline{z}) = \frac{2a}{d\Delta}\{\sin\beta\cosh\alpha[\cos\beta\overline{z}\sinh\alpha\overline{z} + \cos\beta(\overline{z}-1)\sinh\alpha(\overline{z}-1)] -$$

$$-\cos\beta\sinh\alpha[\sin\beta\overline{z}\cosh\alpha\overline{z} + \sin\beta(\overline{z}-1)\cosh\alpha(\overline{z}-1)]\},$$

$$\overline{g}_0(\overline{z}) = \frac{2a}{d\Delta}\{\cos\beta\sinh\alpha[\cos\beta\overline{z}\sinh\alpha\overline{z} + \cos\beta(\overline{z}-1)\sinh\alpha(\overline{z}-1)] +$$

$$+\sin\beta\cosh\alpha[\sin\beta\overline{z}\cosh\alpha\overline{z} + \sin\beta(\overline{z}-1)\cosh\alpha(\overline{z}-1)]\}, \quad (25)$$

where:

$$\Delta = 4[\sinh^2\alpha + \sin^2\beta],$$

$$\alpha^2 = \frac{Re_m(\sqrt{1+\alpha_m^2} - \alpha_m)}{2(1+\alpha_m^2)},$$

$$\beta^2 = \frac{Re_m(\sqrt{1+\alpha_m^2} + \alpha_m)}{2(1+\alpha_m^2)}. \quad (26)$$

This approximation was determined by Pricina, Tigoiu and Cipu in [12].

The system for the one order approximation is:

$$\bar{g}_1'' - \alpha_m \bar{f}_1'' + \bar{A}(\bar{f}_0, \bar{g}_0) = Re_m \bar{f}_1,$$

$$-\bar{f}_1'' - \alpha_m \bar{g}_1'' + \bar{B}(\bar{f}_0, \bar{g}_0) = Re_m \bar{g}_1, \quad (27)$$

where:

$$\bar{A}(\bar{f}_0, \bar{g}_0) = 2\bar{f}_0'\bar{f}_0''\bar{g}_0' + 3\bar{g}_0'^2\bar{g}_0'' + \bar{f}_0'^2\bar{g}_0'',$$

$$\bar{B}(\bar{f}_0, \bar{g}_0) = 2\bar{f}_0'\bar{g}_0'\bar{g}_0'' + 3\bar{f}_0'^2\bar{f}_0'' + \bar{f}_0''\bar{g}_0'^2, \quad (28)$$

with f_0 and g_0 given by the (25) .

The system (27) can be written as:

$$\overline{f}_1'' = -\frac{Re_m}{1+\alpha_m^2}(\alpha_m\overline{f}_1 + \overline{g}_1) + \frac{\alpha_m\bar{A}(z) - \bar{B}(z)}{1+\alpha_m^2},$$

$$\overline{g}_1'' = \frac{Re_m}{1+\alpha_m^2}(\overline{f}_1 - \alpha_m\overline{g}_1) - \frac{\alpha_m\bar{B}(z) + \bar{A}(z)}{1+\alpha_m^2}. \quad (29)$$

The boundary conditions for f_1 and g_1 are as follows:

$$\overline{f}_1(0) = \overline{f}_1(1) = 0,$$
$$\overline{g}_1(0) = \overline{g}_1(1) = 0. \tag{30}$$

If we note:

$$\mathbf{Y}(\bar{z}) = \begin{pmatrix} \overline{f}_1(\bar{z}) \\ \overline{g}_1(\bar{z}) \\ \bar{f}_1'(\bar{z}) \\ \bar{g}_1'(\bar{z}) \end{pmatrix},$$

then the problem is reduced to finding the solution for the following non-homogeneous system:

$$\mathbf{Y}'(\bar{z}) = C\mathbf{Y}(\bar{z}) + \mathbf{D}(\bar{z}), \tag{31}$$

where we denote:

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{Re_m}{1+\alpha_m^2}\alpha_m & -\frac{Re_m}{1+\alpha_m^2} & 0 & 0 \\ \frac{Re_m}{1+\alpha_m^2} & -\frac{Re_m}{1+\alpha_m^2}\alpha_m & 0 & 0 \end{pmatrix}$$

$$\mathbf{D}(\bar{z}) = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha_m \bar{A}(\bar{z}) - \bar{B}(\bar{z})}{1+\alpha_m^2} \\ -\frac{\alpha_m \bar{B}(\bar{z}) + \bar{A}(\bar{z})}{1+\alpha_m^2} \end{pmatrix}. \tag{32}$$

If we denote:

$$\mathbf{Y}(\bar{z}) = \begin{pmatrix} y_1(\bar{z}) \\ y_2(\bar{z}) \\ y_3(\bar{z}) \\ y_4(\bar{z}) \end{pmatrix},$$

the conditions (30) are written as follows:

$$y_1(0) = 0; \; y_2(0) = 0;$$

$$y_1(1) = 0; \; y_2(1) = 0. \tag{33}$$

The general solution of the non-homogeneous system (31), has the form:

$$\mathbf{Y}(\bar{z}) = C_1\boldsymbol{\varphi}_1(\bar{z}) + C_2\boldsymbol{\varphi}_2(\bar{z}) + C_3\boldsymbol{\varphi}_3(\bar{z}) + C_4\boldsymbol{\varphi}_4(\bar{z}) + \boldsymbol{\varphi}_p(\bar{z}). \tag{34}$$

In this expression $\varphi_p(\bar{z})$ is the particular solution of the non-homogeneous system and $\varphi_1(\bar{z}), \varphi_2(\bar{z}), \varphi_3(\bar{z}), \varphi_4(\bar{z})$ are the solutions of the homogeneous associated system:

$$\mathbf{Y}' = C\mathbf{Y}. \tag{35}$$

The solutions of the homogeneous system verify the following conditions:

$$\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{36}$$

The particular solution $\varphi_p(\bar{z})$ of the non-homogeneous system (31), verifies the following condition:

$$\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{37}$$

The components of the fundamental solution and the components of the particular solution, are written in the following way $\varphi_1 \equiv \varphi_{1i}, \varphi_2 \equiv \varphi_{2j}, \varphi_3 \equiv \varphi_{3k}, \varphi_4 \equiv \varphi_{4l}, \varphi_p \equiv \varphi_{pm}$, with $i,j,k,l,m \in \{1,2,3,4\}$.

We determine the constants: C_1, C_2, C_3, C_4 so that $Y(\bar{z})$ is a solution of the system (31).

Keeping in mind that the solution for $Y(\bar{z})$ given by (34) and the conditions :(33),(36), we obtained: $C_1 = C_2 = 0$ and the following system for the constants C_3 and C_4:

$$C_3\varphi_{31}(1) + C_4\varphi_{41}(1) + \varphi_{p1}(1) = 0,$$

$$C_3\varphi_{32}(1) + C_4\varphi_{42}(1) + \varphi_{p2}(1) = 0. \tag{38}$$

The solutions of this system are:

$$C_3 = \frac{\varphi_{p2}(1)\varphi_{41}(1) - \varphi_{p1}(1)\varphi_{42}(1)}{\varphi_{31}(1)\varphi_{42}(1) - \varphi_{41}(1)\varphi_{32}(1)},$$

$$C_4 = \frac{\varphi_{p1}(1)\varphi_{32}(1) - \varphi_{p2}(1)\varphi_{31}(1)}{\varphi_{31}(1)\varphi_{42}(1) - \varphi_{41}(1)\varphi_{32}(1)}. \quad (39)$$

For the solutions of the homogeneous system (35), we get:

$$\varphi_{31}(\bar{z}) = \frac{1}{\alpha^2+\beta^2}(\alpha\cos\beta\bar{z}\sinh\alpha\bar{z} + \beta\sin\beta\bar{z}\cosh\alpha\bar{z}),$$

$$\varphi_{32}(\bar{z}) = \frac{1}{\alpha^2+\beta^2}(-\beta\cos\beta\bar{z}\sinh\alpha\bar{z} + \alpha\sin\beta\bar{z}\cosh\alpha\bar{z}),$$

$$\varphi_{41}(\bar{z}) = \frac{1}{\alpha^2+\beta^2}(\beta\cos\beta\bar{z}\sinh\alpha\bar{z} - \alpha\sin\beta\bar{z}\cosh\alpha\bar{z}),$$

$$\varphi_{42}(\bar{z}) = \frac{1}{\alpha^2+\beta^2}(\alpha\cos\beta\bar{z}\sinh\alpha\bar{z} + \beta\sin\beta\bar{z}\cosh\alpha\bar{z}). \quad (40)$$

Therefore, the solution of the non-homogeneous system (31), has the following form:

$$\mathbf{Y}(\bar{z}) = C_3\boldsymbol{\varphi}_3(\bar{z}) + C_4\boldsymbol{\varphi}_4(\bar{z}) + \boldsymbol{\varphi}_p(\bar{z}), \quad (41)$$

Where the φ_3, φ_4 are given by formulas (40) and the particular solution φ_p is determined numerically by using a Runge Kutta method. In this way, we determined the $Y(\overline{z})$ solution, as well as the first order terms: f_1 and g_1.

In conclusion, we determined the one order approximation of the bilocal problem (20)-(21):

$$\bar{f}(\bar{z}) = \bar{f}_0(\bar{z}) + \varepsilon\bar{f}_1(\bar{z}),$$

$$\bar{g}(\bar{z}) = \bar{g}_0(\bar{z}) + \varepsilon\bar{g}_1(\bar{z}). \quad (42)$$

4. NUMERICAL STUDY. COMPARISONS.

For the third grade fluids we used the following restrictions: (see V. Tigoiu [11])

$\mu \geq 0,\ \alpha_1 \geq 0,\ \beta_1 \leq 0,\ \beta_1 + 2(\beta_2+\beta_3) \geq 0.$

In the graphics, we used the values:

$p = 1000 kg\cdot m^{-3}; \Omega = 40 rad\cdot s^{-1}, d = 0.015m; a = d$.

We can consider the following three sets of values for the constitutive modules:

Set I:

$$\mu = 8kg \cdot m^{-1} \cdot s^{-1}; \alpha_1 = 0.225\, kg \cdot m^{-1}; \beta_1 = 0.000625\ kg \cdot m^{-1} \cdot s \Rightarrow \mathrm{Re}_m = \alpha_m = 1(\mathrm{Re} = \theta = 1.125),$$

Set II:

$$\mu = 80, \alpha_1 = 2.25; \beta_1 = -0.00625 \Rightarrow \mathrm{Re}_m = 0.1; \alpha_m = 1; (\mathrm{Re} = 0.1125; \theta = 1.125,$$

Set III:

$$\mu = 0.8; \alpha_1 = 0.0225; \beta_1 = -0.0000625 \Rightarrow \mathrm{Re}_m = 10; \alpha_m = 1; (\mathrm{Re} = 11.25; \theta = 1.125)$$

In Fig.2. and Fig.3 are represented the first order approximations: $\overline{f_0}(\overline{z}) + \varepsilon \overline{f_1}$ and $\overline{g_0}(\overline{z}) + \varepsilon \overline{g_1} - \frac{a}{d}(\overline{z} - 0.5)$, for the above three sets of values and ε = 0.1. It can be noticed that the values for $\overline{f}$ *and* $\overline{g}$ are influenced by Re_m, which means that the amplitude is increasing if Re_m increasing as well.

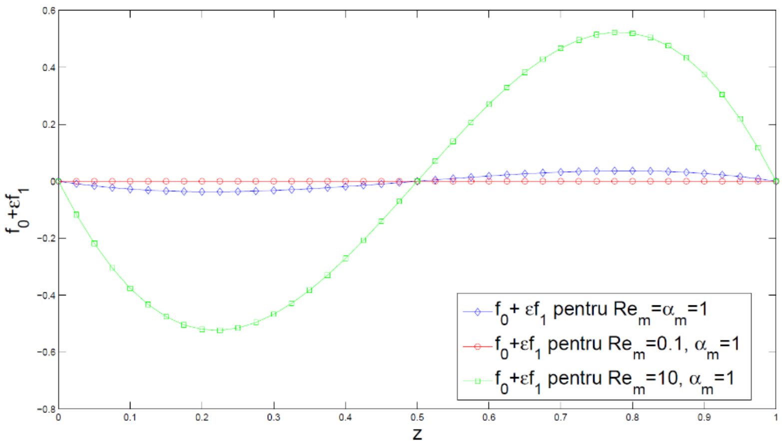

Figure 2. The approximation $\overline{f_0}(\overline{z}) + \varepsilon \overline{f_1}$ for the three sets of values.

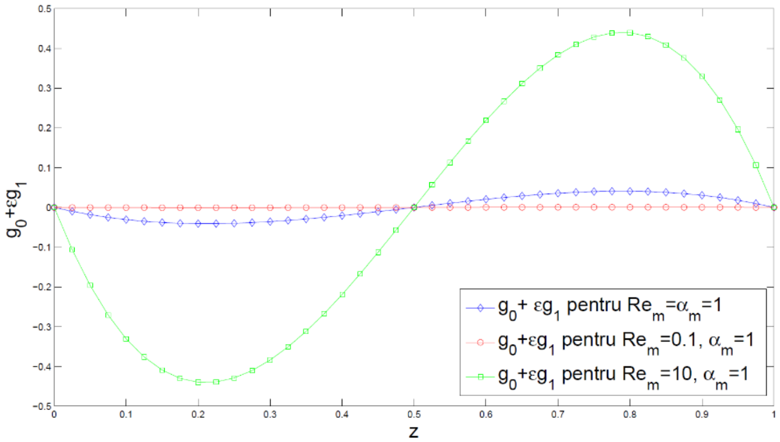

Figure 3. The $\overline{g}_0(\overline{z})+\varepsilon\overline{g}_1-\frac{a}{d}(\overline{z}-0.5)$ approximation for the three sets of values.

In Fig.4-Fig.5, we compare the solutions for the second grade fluid (obtained by Rajagopal in [5)]with the zero order approximations for the third grade fluid. We notice that β1 influences quite a little the values of *f* and in small way the values of *g* , for all three sets of data.

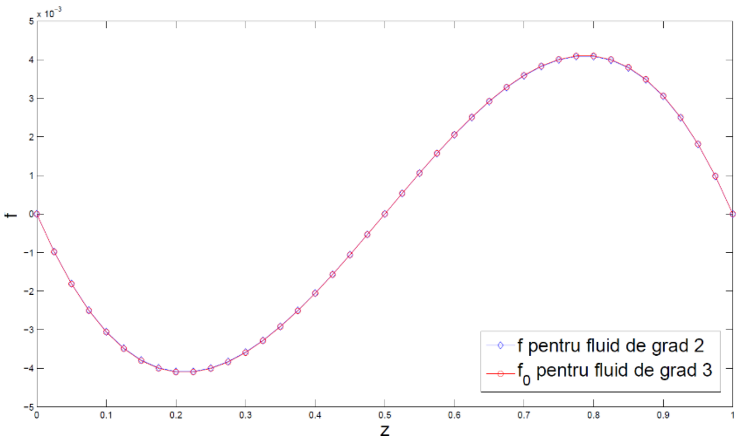

Figure 4. Comparisons between $\overline{f}(\overline{z})$ for the (Re = θ = 1.125) second grade fluid and $\overline{f}_0(\overline{z})$ for the third grade fluid in the first set of data.

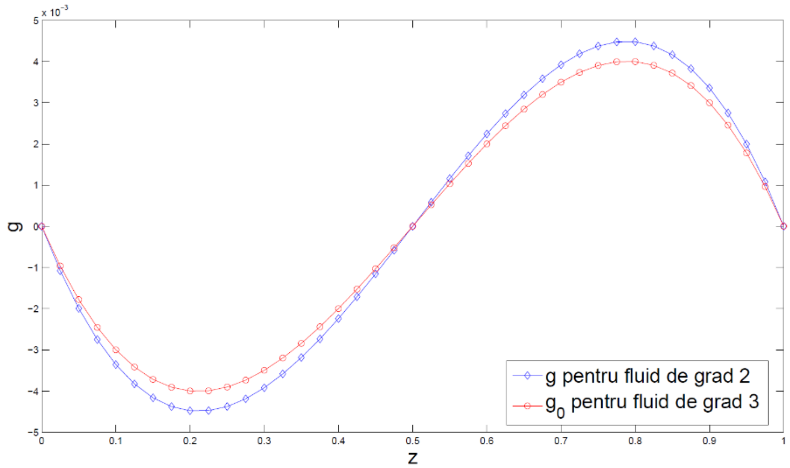

Figure 5. Comparisons between: $\overline{g}(\overline{z})-\frac{a}{d}(\overline{z}-0.5)$ for the second grade fluid and

$\overline{g}_0(\overline{z})-\frac{a}{d}(\overline{z}-0.5)$ for the third grade fluid in the first set of value.

In Fig.6-Fig.7, we compare the numerical solutions of "main system", that have been determinated with a method with a finite difference, with zero order and one order approximations. Numerical solution is very close to one order approximation while zero order approximation is very close to the other two only in the case of $\overline{f}$.

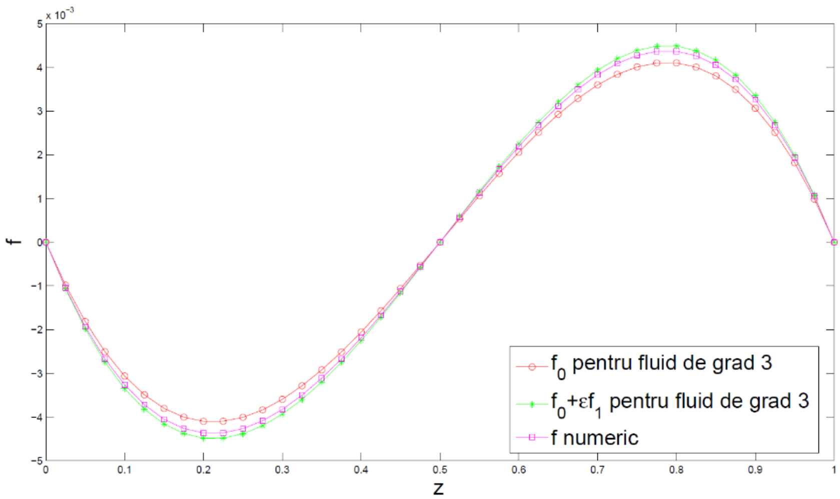

Figure 6. Comparison between $\overline{f_0}(\overline{z}), \overline{f_0}(\overline{z})+\varepsilon\overline{f_1}(\overline{z})$ and numerical solution for the third grade fluid, for first set and $\varepsilon = 0.1$.

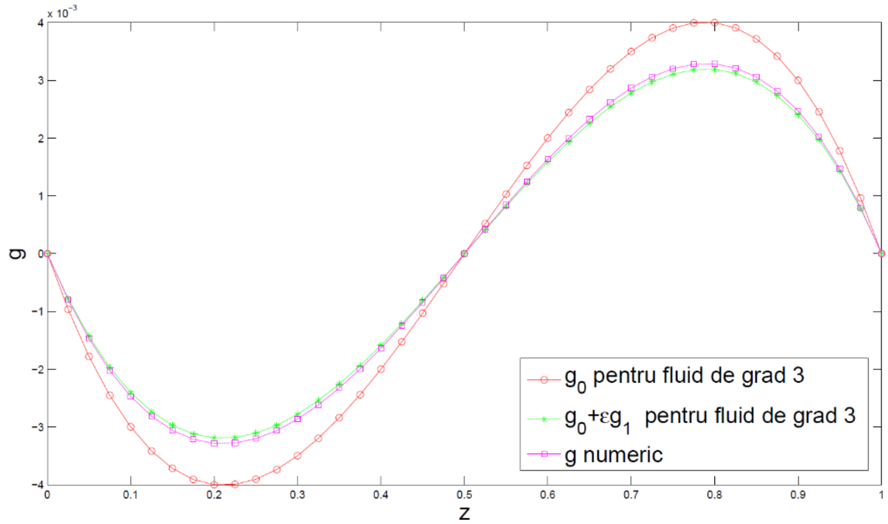

Figure 7. Comparison between $\overline{g}_0(\overline{z})-\frac{a}{d}(\overline{z}-0.5), \overline{g}_0(\overline{z})+\varepsilon\overline{g}_1(\overline{z})-\frac{a}{d}(\overline{z}-0.5)$ and numerical solution for third grade fluid, for first set of value and $\varepsilon = 0.1$.

In Fig.8.i s represented the velocity field for $\mathrm{Re}_m = \alpha_m = 1$.

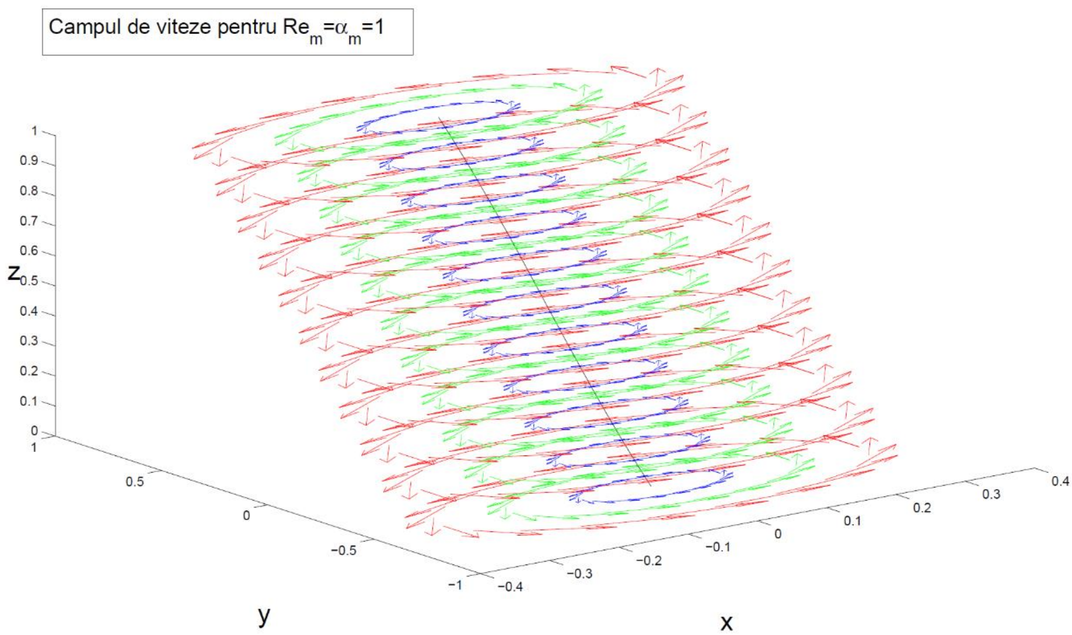

Figure 8. The Velocity field for $\mathrm{Re}_m = \alpha_m = 1$.

In Fig.9. are represented the streamlines for $\mathrm{Re}_m = \alpha_m = 1$.

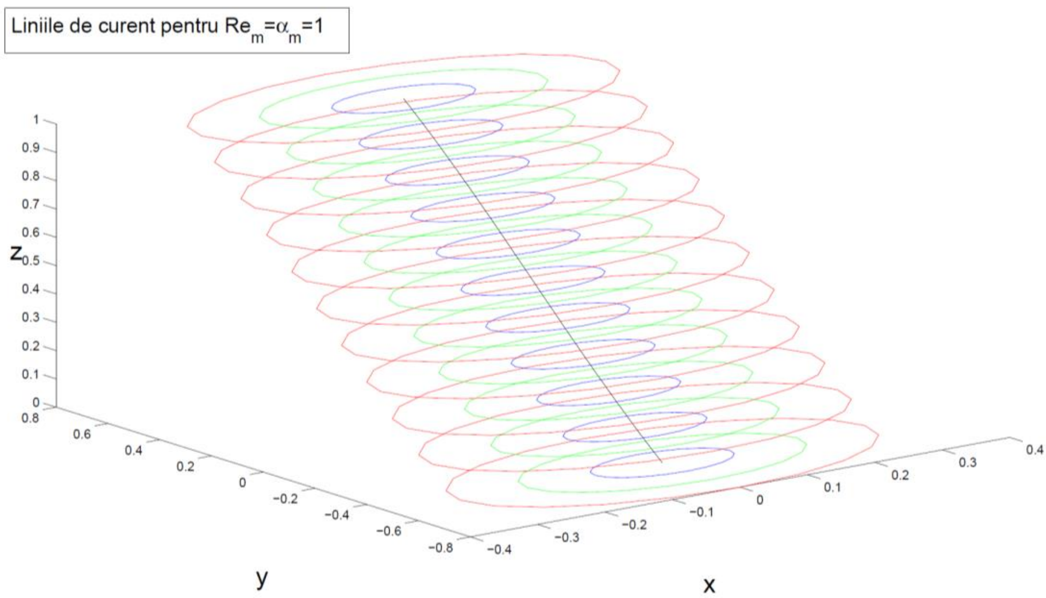

Figure 9. The streamlines for $\mathrm{Re}_m = \alpha_m = 1$.

We observe that the motion occurring in parallel horizontal planes and the streamlines are circles.

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